ANALYSIS OF SEVERAL VARIABLES MIDTERM EXAMINATION

Total marks: 50

Attempt all questions. If you are using a theorem done in class in your answer, please quote it in full.

Time: 3 hours

(1) Give an example of a function all of whose directional derivatives at a point exist and are zero, but the function is not continuous at that point. (8 marks)

(2) Suppose a hillside is given by $z = f(x, y) = \frac{40}{4+x^2+3y^2}$, where $(x, y) \in U \subset \mathbb{R}^2$ ($z = f(x, y)$ is the height at the point (x, y)). Find a vector tangent to the curve of steepest ascent on the hill at the point $(1, 1, 5)$. (10 marks)

(3) Consider the function $f(x, y) = 2x^4 - 3x^2y + y^2$. Show that the origin is a critical point of f and that, restricting f to any line through the origin, the origin is a local minimum point. Is the origin a local minimum point of f? (5+5=10 marks)

(4) What is the smallest number A such that **any** two squares S_1 and S_2 of total area 1 (unit) can be put disjointly into a rectangle of area A (units)? The two squares can be put disjointly means their interiors do not intersect. (10 marks)

(5) Let $O(n)$ be the subset $\{A \in M_n(\mathbb{R}) | A^T A = I_n\}$ of $\mathbb{R}^{n^2}$. Show that $O(n)$ is a compact subset of $\mathbb{R}^{n^2}$. Prove that $O(n)$ is a smooth manifold and find the dimension of $O(n)$. (2+10 =12 marks)